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REMARKS ON POSITIVE MAPS ON SELFDUAL CONES

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ここではヒルベルト空間における selfdual cone を保存する意味での正値写像および作用素の順序 ($\trianglelefteq$) に関する基本的な性質を考える。内容は [MI] を部分的に含む。

§1. INTRODUCTION

Let $\mathcal{H}$ be a separable complex Hilbert space with an inner product (,). A convex cone $\mathcal{H}^+$ in $\mathcal{H}$ is said to be selfdual if $\mathcal{H}^+ = \{\xi \in \mathcal{H} | (\xi, \eta) \geq 0 \ \forall \eta \in \mathcal{H}^+\}$. The set of all bounded operators is denoted by $L(\mathcal{H})$. For a fixed selfdual cone $\mathcal{H}^+$, we shall write

$$A \trianglelefteq B \quad \text{if} \quad (B-A)(\mathcal{H}^+) \subset \mathcal{H}^+, A, B \in L(\mathcal{H}).$$

Since $\mathcal{H}$ is algebraically spanned by $\mathcal{H}^+$, the relation '$\trianglelefteq$ ' defines the partial order on $L(\mathcal{H})$.

Recall a selfdual cone associated with a standard von Neumann algebra in the sense of Haagerup [H], which appears in the form $(\mathcal{M}, \mathcal{H}, J, \mathcal{H}^+)$ where $\mathcal{M}$ is a von Neumann algebra on $\mathcal{H}$ and J is an isometric involution related to a selfdual cone$\mathcal{H}^+$ in $\mathcal{H}$. For example, $\ell^{2+} = \{\xi = \{\lambda_n\} | \lambda_n \geq 0\}$ is a selfdual cone associated with an abelian standard von Neumann algebra ℓ^∞. Then, for $A = (\lambda_{ij}) \in L(\ell^2)$, $A \trianglerighteq O$ if and only if $\lambda_{ij} \geq 0$ for $i, j = 1, 2, \cdots$.

Moreover, suppose that $(\mathcal{H}, \mathcal{H}_n^+, n \in \mathbf{N})$ and $(\tilde{\mathcal{H}}, \tilde{\mathcal{H}}_n^+, n \in \mathbf{N})$ are matrix ordered Hilbert spaces. Here $\mathcal{H}_n^+$ denotes a selfdual cone in $\mathcal{H}_n = M_n(\mathcal{H})$. A linear map A of $\mathcal{H}$ into $\tilde{\mathcal{H}}$ is said to be n-positive(resp. n-co-positive) when the multiplicity map $A_n(= A \otimes \mathrm{id}_n)$ satisfies $A_n \mathcal{H}_n^+ \subset \tilde{\mathcal{H}}_n^+$ (resp. ${}^t(A_n \mathcal{H}_n^+) \subset \tilde{\mathcal{H}}_n^+$). Here ${}^t(\cdot)$ denotes a set of all transposed matrices. When A is n-positive(resp. n-co-positive) for all

$n \in \mathbb{N}$, A is said to be completely positive(resp. completely co-positive). Put, for $A \in L(\mathcal{H})$

$$\hat{A}\xi = AJAJ\xi, \quad \xi \in \mathcal{H}.$$

It is known that if, in a matrix ordered standard form $(\mathcal{M}, \mathcal{H}, \mathcal{H}_n^+)$ as introduced in [**SW2**], $A \in \mathcal{M}$ then $\hat{A}$ is completely positive, and we shall write $\hat{A} \trianglerighteq_{cp} O$.

§2. Positive maps associated with selfdual cones

We obtain the following proposition for a general selfdual cone in a finite dimensional Hilbert space. In particular, when $\mathcal{H}^+$ is associated with an abelian von Neumann algebra, that is, a matrix is entrywise positive, it is known as the Peron theorem(see, example [**HJ**, Corollary 8.2.6]).

(2.1). *Let $\mathcal{H}$ be an n-dimensional Hilbert space with a selfdual cone $\mathcal{H}^+$. If A is an injective linear operator on $\mathcal{H}$ satisfying $A \trianglerighteq O$, then there exist a number $\lambda > 0$ and a non-zero element $\xi_0 \in \mathcal{H}^+$ such that $A\xi_0 = \lambda\xi_0$.*

Proof. Put

$$\mathcal{V} = \mathrm{co}\{\xi \in \mathcal{H}^+ | \parallel \xi \parallel = 1\},$$

where co denotes the convex hull. Consider the map r defined by

$$r(\xi) = \frac{A\xi}{\parallel A\xi \parallel}, \xi \in \mathcal{V}.$$

By assumption r maps $\mathcal{V}$ to itself. Note that $0 \notin \mathcal{V}$. Because, by the Carathéodory theorem(see, for example [**La**, Theorem 2.23]) any element $\xi \in \mathcal{V}$ can be expressed as

$$\xi = \lambda_1\xi_1 + \cdots + \lambda_s\xi_s,$$

where $\lambda_1, \cdots, \lambda_s > 0, \xi_1, \cdots, \xi_s \in \mathcal{H}^+$ with $\parallel \xi_1 \parallel = \cdots = \parallel \xi_s \parallel = 1$ and $1 \leq s \leq n+1$. It follows that $\xi \geq \lambda_1\xi_1(\mathcal{H}^+)$, and so $\parallel \xi \parallel \geq \parallel \lambda_1\xi_1 \parallel = |\lambda_1| > 0$. Since a convex hull of a compact set is compact [**La**, Theorem 2.30], it follows from Schauder's fixed point theorem [**Sd**, Satz I] that there exists an element $\xi_0 \in \mathcal{V}$ satisfying $r(\xi_0) = \xi_0$. Hence $A\xi_0 = \parallel A\xi_0 \parallel \xi_0$. □

The following fundamental proposition is valid for a general selfdual cone. It says that the order '$\trianglelefteq$' is different from the usual order '$\leq$' based on positivity of hermitian operators in point of compatibility with product.

(2.2). (cf. [IM, Proposition 1]) *Let $\mathcal{H}$ be a Hilbert space with a selfdual cone $\mathcal{H}^+$. Then for bounded operators on $\mathcal{H}$ we have the following properties:*

(1) *If $O \trianglelefteq A_1 \trianglelefteq B_1$ and $O \trianglelefteq A_2 \trianglelefteq B_2$, then $O \trianglelefteq A_1A_2 \trianglelefteq B_1B_2$. In particular, if $O \trianglelefteq A \trianglelefteq B$, then $A^n \trianglelefteq B^n$ for every natural number n.*

(2) *If $O \trianglelefteq A \trianglelefteq B$, then $O \trianglelefteq A^* \trianglelefteq B^*$.*

(3) *If $A, A^{-1}, B, B^{-1} \trianglerighteq O$ and $A \trianglelefteq B$, then $B^{-1} \trianglelefteq A^{-1}$.*

(4) *If $O \trianglelefteq A \trianglelefteq B$, then $\| A \| \leq \| B \|$.*

Proof. We sketch a proof which is similar to [IM].

(1) By assumption $A_i(\mathcal{H}^+) \subset \mathcal{H}^+$ and $(B_i - A_i)(\mathcal{H}^+) \subset \mathcal{H}^+$ hold for $i = 1,2$. Since $B_1B_2 - A_1A_2 = B_1(B_2 - A_2) + (B_1 - A_1)A_2$, we obtain the desired inequality.

(2) Let $A(\mathcal{H}^+) \subset \mathcal{H}^+$. Then we have $(A^*\xi, \eta) = (\xi, A\eta) \geq 0$ for all $\xi, \eta \in \mathcal{H}^+$. The selfduality of $\mathcal{H}^+$ shows that $A^* \trianglerighteq O$. Exchanging the role of A and $B - A$ we obtain the desired property.

(3) If $A \trianglelefteq B$, then $B^{-1} = A^{-1}AB^{-1} \trianglelefteq A^{-1}BB^{-1} = A^{-1}$ from (1).

(4) For $A \trianglerighteq O$, put $\| A \|_+ = \sup\{\| A\xi \|; \| \xi \| \leq 1, \xi \in \mathcal{H}^+\}$. Suppose $O \trianglelefteq A \trianglelefteq B$. Note that if $\eta - \xi \in \mathcal{H}^+$ for $\xi, \eta \in \mathcal{H}^+$, then $\| \xi \| \leq \| \eta \|$. Since $\| A \|_+ \leq \| B \|_+$, it suffices to show $\| \cdot \|_+ = \| \cdot \|$. It is known that any element $\xi \in \mathcal{H}$ can be written as $\xi = \xi_1 - \xi_2 + i(\xi_3 - \xi_4), \xi_1 \perp \xi_2, \xi_3 \perp \xi_4$, for some $\xi_i \in \mathcal{H}^+$. Then $\| \xi \|^2 = \sum_{i=1}^4 \| \xi_i \|^2$. Noticing that $A \trianglerighteq O$, we see that

$$\begin{aligned} \| A\xi \|^2 &= \sum_{i=1}^4 \| A\xi_i \|^2 - 2(A\xi_1, A\xi_2) - 2(A\xi_3, A\xi_4) \\ &\leq \| A(\xi_1 + \xi_2) \|^2 + \| A(\xi_3 + \xi_4) \|^2 \leq \| A \|_+^2 \| \xi \|^2 . \end{aligned}$$

It follows that $\| A \| \leq \| A \|_+$. The converse inequality is trivial. □

(2.3). *Let $(\mathcal{M}, \mathcal{H}, J, \mathcal{H}^+)$ be a standard form of a von Neumann algebra. For a selfadjoint element $A \in \mathcal{M} \cup \mathcal{M}'$, the following conditions are equivalent:*

(1) *$A \trianglerighteq O$.*

(2) *$A \in Z(\mathcal{M})$ and $A \geq O$.*

Proof. (1)$\Rightarrow$ (2): Since $A \trianglerighteq O$ if and only if $JAJ \trianglerighteq O$, it suffices to investigate the case $A \in \mathcal{M}$. Suppose $A \trianglerighteq O, A \in \mathcal{M}$. Since any element of $\mathcal{H}$ can be written as $\xi + i\eta$ with $J\xi = \xi, J\eta = \eta$, it follows that for such elements ξ, η

$$JAJ(\xi + i\eta) = JA(\xi - i\eta) = JA\xi + iJA\eta = A(\xi + i\eta).$$

Hence $A \in Z(\mathcal{M})$ and $A^* = JAJ = A$. Choose an arbitrary element $\xi \in \mathcal{H}$. Then one can write as $\xi = \xi_1 - \xi_2 + i(\xi_3 - \xi_4), \xi_i \in \mathcal{H}^+$ such that $\mathcal{M}\xi_1 \perp \mathcal{M}\xi_2, \mathcal{M}\xi_3 \perp \mathcal{M}\xi_4$. We then have

$$\begin{aligned}(A\xi, \xi) &= (A\xi_1 - A\xi_2 + i(A\xi_3 - A\xi_4), \xi_1 - \xi_2 + i(\xi_3 - \xi_4)) \\ &= \sum_{i=1}^{4}(A\xi_i, \xi_i) \trianglerighteq O\end{aligned}$$

because $(A\xi_1, \xi_2) = (A\xi_3, \xi_4) = 0$ and $((A(\xi_1 - \xi_2), \xi_3 - \xi_4)$ is a real number. Hence $A \geq O$.

(2)$\Rightarrow$ (1): It is immediate. $\square$

(2.4). *Suppose that $A \in L(\mathcal{H})^+$ has a closed range in which $A\mathcal{H}^+$ is a selfdual cone. Then we obtain the following properties:*

(1) *Under the condition that $\mathcal{H}^+$ is a facially homogeneous selfdual cone in $\mathcal{H}$, if $A \trianglerighteq O$, then for all $\lambda \in \mathbb{R}$, $A^\lambda \trianglerighteq O$.*

(2) *For a matrix ordered standard form $(\mathcal{M}, \mathcal{H}, \mathcal{H}_n^+)$, if $A \trianglerighteq O$ and the support projection of A is completely positive, then for all $\lambda \in \mathbb{R}$, $A^\lambda \trianglerighteq_{cp} O$.*

Here the inverse for a not invertible A is taken as reduced by the support projection of A.

Proof. (1) Let P denote the support projection of A. By assumption we obtain that $P \trianglerighteq O$ and $P\mathcal{H}^+ = A\mathcal{H}^+$. Hence, by [**I**, Proposition II.1.6], $P\mathcal{H}^+$ is facially homogeneous. Since $A = PA = AP$ and PA maps $P\mathcal{H}^+$ onto itself, it follows from [**I**, Corollary II.3.2] that there exists a derivation $\delta \in D(P\mathcal{H}^+)^+$ such that $PA|_{P\mathcal{H}} = e^\delta$. Hence

$$A^\lambda = Pe^{\lambda\delta}P \trianglerighteq O$$

for every real number λ.

(2) Put $\mathcal{N} = P\mathcal{M}|_{P\mathcal{H}}$. Since P is completely positive, we see from [**MN**, Lemma 3] that $(\mathcal{N}, P\mathcal{H}, P_n\mathcal{H}_n^+)$ is a matrix ordered standard form. It follows

from [C, Theorem 3.3] that there exists an element $B \in \mathcal{N}^+$ such that $PA = BJ_{P\mathcal{H}^+}BJ_{P\mathcal{H}^+}P$. Hence

$$A^\lambda = B^\lambda J_{P\mathcal{H}^+}B^\lambda J_{P\mathcal{H}^+}P \trianglerighteq_{cp} O$$

for every real number λ. $\square$

A simple counter-example can show that it is essential in the above proposition for $A\mathcal{H}^+$ to be selfdual. In fact, we obtain the following remark:

Remark. In the case $\mathbf{C}^{n+}$(non-negative entries), a necessary and sufficient condition for $A \in M_n^+$ to enjoy $A\mathbf{C}^{n+} = \mathbf{C}^{n+}$ is that A is a non-singular positive definite diagonal matrix. We obtain the following facts:

(1) In the case $\mathbf{C}^{n+}$, if $A \in M_n^+$ and $A \trianglerighteq O$, then there exists a real number $s \geq 1$ such that $A^\lambda \trianglerighteq O$ for all $\lambda \in [s, +\infty)$.

(2) In the case $\mathbf{C}^{n+}$, if $A \in M_n^+, A \trianglerighteq O, \det A \neq 0$ and $A\mathbf{C}^{n+} \subsetneq \mathbf{C}^{n+}$, then there exists a real number $s' < 0$ such that $A^\lambda \not\trianglerighteq O$ for all $\lambda \in (-\infty, s']$.

Indeed, let $A \in M_n$ be entrywise positive and positive semi-definite. We may assume $\| A \|= 1$. Let $1, a_1, \cdots, a_m, 0 \leq m \leq n-1$, be distinct eigenvalues of A. Since A can be diagonalized by a real orthogonal matrix, each entry of A^λ is written in the form

$$f(\lambda) = \alpha_0 + \alpha_1 a_1^\lambda + \cdots + \alpha_m a_m^\lambda$$

for some real numbers α_k. Then α_0 must be positive, since $A^n \trianglerighteq O$ for all $n \in \mathbf{N}$ by (2.2) (1) and $0 \leq a_k < 1, 1 \leq k \leq m$. From the continuity of the function we can find a number $s \geq 1$ such that $f(\lambda) > 0$ for all $\lambda \geq s$. So (1) holds. Suppose, in addition, that A is non-singular and $A\mathbf{C}^{n+} \subsetneq \mathbf{C}^{n+}$. If $A^{-\lambda_0} \trianglerighteq O$ for some $\lambda_0 > 0$, then $A^{-\ell\lambda_0} \trianglerighteq O$ for all$\ell \in \mathbf{N}$. From (1), $A^{\ell\lambda_0} \trianglerighteq O$ for a large $\ell \in \mathbf{N}$. This implies that $A^{\ell\lambda_0}$ is diagonal, and so is A, a contradiction. Therefore, (2) holds.

(2.5). *For a matrix ordered standard form $(\mathcal{M}, \mathcal{H}, \mathcal{H}_n^+)$, suppose that $A \in L(\mathcal{H})$, and $B \in \mathcal{M}$ is an injective operator with a dense range. Then, $O \trianglelefteq A \trianglelefteq \hat{B}$ if and only if there exists an element $C \in Z(\mathcal{M})$ with $O \leq C \leq I$ such that $A = C\hat{B}$. In particular, if $\mathcal{M}$ is a factor, then one can choose a scalar λ with $0 \leq \lambda \leq 1$ such that $A = \lambda\hat{B}$.*

Proof. Consider the polar decomposition $B = U|B|$ of B. By assumption U is a unitary element of $\mathcal{M}$, and so $\hat{U} \trianglerighteq O$ and $\hat{U}^* \trianglerighteq O$ by (2.2). Hence we may assume B to be positive semi-definite. Let $B = \int_0^{\|B\|} \lambda dE_\lambda$ be a spectral decomposition of B. Put $P_n = \int_{\frac{1}{n}}^{\|B\|} dE_\lambda$ for $n \in \mathbb{N}$. Then one sees that $\hat{P}_n \nearrow I$ and $\hat{P}_n A \hat{P}_n \trianglelefteq \hat{P}_n \hat{B} \hat{P}_n$ by (2.2). Since $\hat{P}_n \hat{B} \hat{P}_n$ is invertible on $\hat{P}_n \mathcal{H}$, where the inverse shall be denoted by $(\hat{P}_n \hat{B} \hat{P}_n)^{-1}$, we have

$$O \trianglelefteq \hat{P}_n A \hat{P}_n (\hat{P}_n \hat{B} \hat{P}_n)^{-1} \trianglelefteq \hat{P}_n.$$

There then exists an element c_n in an order ideal $Z_{\hat{P}_n \mathcal{H}^+}$ of a selfdual cone $\hat{P}\mathcal{H}^+$ with $\| c_n \| \leq 1$ such that $\hat{P}_n A \hat{P}_n (\hat{P}_n \hat{B} \hat{P}_n)^{-1} \xi = c_n \xi$ for all $\xi \in \hat{P}_n \mathcal{H}$. By [**I**, Theorem VI.1,2 3)] we obtain that $c_n \in Z(\hat{P}_n \mathcal{M}|_{\hat{P}_n \mathcal{H}})^+$. Since $\hat{P}_n Z(\mathcal{M}) \hat{P}_n = Z(\hat{P}_n \mathcal{M} \hat{P}_n)$, we can find an element $C_n \in Z(\mathcal{M})$ such that $c_n \xi = \hat{P}_n C_n \hat{P}_n \xi$ for all $\xi \in \hat{P}_n \mathcal{H}$. Since $P_n B = B P_n, n \in \mathbb{N}$, we have

$$\begin{aligned} \hat{P}_{n+1} C_{n+1} \hat{P}_{n+1} \xi &= \hat{P}_{n+1} A \hat{P}_{n+1} (\hat{P}_{n+1} \hat{B} \hat{P}_{n+1})^{-1} \hat{P}_n \xi \\ &= \hat{P}_{n+1} A \hat{P}_n (\hat{P}_n \hat{B} \hat{P}_n)^{-1} \xi = \hat{P}_n C_n \hat{P}_n \xi \end{aligned}$$

for all $\xi \in \hat{P}_n \mathcal{H}$. Since $\{\hat{P}_n C_n \hat{P}_n\}$ is a bounded sequence, one can define

$$C\xi = \lim_{n \to \infty} \hat{P}_n C_n \hat{P}_n \xi, \ \xi \in \mathcal{H}.$$

Thus $C \in Z(\mathcal{M}), O \leq C \leq I$ and we get

$$\begin{aligned} A &= \text{s-} \lim_{n \to \infty} \hat{P}_n A \hat{P}_n \\ &= \text{s-} \lim_{n \to \infty} \hat{P}_n C_n \hat{P}_n A \hat{P}_n \\ &= C\hat{B}. \end{aligned}$$

The converse implication is immediate. Indeed, if $C \in Z(\mathcal{M})$ with $O \leq C \leq I$, then $I - C \geq O$, and so$I - C \trianglerighteq O$. Hence $\hat{B} - C\hat{B} = (I - C)\hat{B} \trianglerighteq O$. This completes the proof. □

§3. Complete order of operators

Consider two matrix ordered standard forms ($\mathcal{M}^{(1)}$, $\mathcal{H}^{(1)}$, $\mathcal{H}_n^{(1)+}$) and ($\mathcal{M}^{(2)}$, $\mathcal{H}^{(2)}$, $\mathcal{H}_n^{(2)+}$) with respective canonical involutions $J^{(1)}$ and $J^{(2)}$. For an arbitrary element $\xi \in \mathcal{H}^{(1)}$, let R_ξ be a right slice map of $\mathcal{H}^{(1)} \otimes \mathcal{H}^{(2)}$ into $\mathcal{H}^{(2)}$ such that

$$R_\xi(\xi' \otimes \eta') = (\xi', \xi)\eta', \xi' \in \mathcal{H}^{(1)}, \eta' \in \mathcal{H}^{(2)}.$$

For any element $x \in \mathcal{H}^{(1)} \otimes \mathcal{H}^{(2)}$, we put

$$r(x)\xi = R_{J^{(1)}\xi}(x), \xi \in \mathcal{H}^{(1)}.$$

Then $r(x)$ is a map of Hilbert-Schmidt class of $\mathcal{H}^{(1)}$ to $\mathcal{H}^{(2)}$. A set of all maps of Hilbert-Schmidt class of $\mathcal{H}^{(1)}$ to $\mathcal{H}^{(2)}$ is denoted by $HS(\mathcal{H}^{(1)}, \mathcal{H}^{(2)})$. A set of all completely positive maps of $(\mathcal{H}^{(1)}, \mathcal{H}_n^{(1)+'})$ to $(\mathcal{H}^{(2)}, \mathcal{H}_n^{(2)+})$ in $HS(\mathcal{H}^{(1)}, \mathcal{H}^{(2)})$ is denoted by $CPHS(\mathcal{H}^{(1)+'}, \mathcal{H}^{(2)+})$. Here $\mathcal{H}_n^{(1)+'}, n \in \mathbf{N}$, means a family of the selfdual cones associated with $\mathcal{M}^{(1)'}$, that is $\mathcal{H}_n^{(1)+'} = \{{}^t[\xi_{ij}]_{i,j=1}^n | [\xi_{ij}]_{i,j=1}^n \in \mathcal{H}_n^{(1)+}\}$. We shall write $\mathcal{H}^{(1)+} \otimes \mathcal{H}^{(2)+}$ for a selfdual cone associated with a von Neumann tensor product $\mathcal{M}^{(1)} \otimes \mathcal{M}^{(2)}$. It was shown in **[MT, SW1]** that

$$\mathcal{H}^{(1)+} \otimes \mathcal{H}^{(2)+} = \{x \in \mathcal{H}^{(1)} \otimes \mathcal{H}^{(2)} | r(x) \in CPHS(\mathcal{H}^{(1)+'}, \mathcal{H}^{(2)+})\}.$$

Thus

$$r: \mathcal{H}^{(1)} \otimes \mathcal{H}^{(2)} \to HS(\mathcal{H}^{(1)}, \mathcal{H}^{(2)})$$

is an isometry mapping $\mathcal{H}^{(1)+} \otimes \mathcal{H}^{(2)+}$ onto $CPHS(\mathcal{H}^{(1)+'}, \mathcal{H}^{(2)+})$.

Indeed, r is isometric. Suppose that $HS(\mathcal{H}^{(1)}, \mathcal{H}^{(2)})$ has an inner product

$$\langle A, B\rangle = \sum_{k=1}^{\infty}(Ae_k, Be_k),$$

where $\{e_k\}$ is a complete orthogonal basis of $\mathcal{H}^{(1)}$. Noticing that $\{J^{(1)}e_k\}$ is a complete orthogonal basis of $\mathcal{H}^{(1)}$, we obtain for a complete orthogonal basis $\{f_k\}$

$$\begin{aligned}\langle r(J^{(1)}e_i \otimes f_j), & r(J^{(1)}e_{i'} \otimes f_{j'})\rangle \\ &= \sum_{k=1}^{\infty}(r(J^{(1)}e_i \otimes f_j)(e_k), r(J^{(1)}e_{i'} \otimes f_{j'})(e_k)) \\ &= \sum_{k=1}^{\infty}(R_{J^{(1)}e_k}(J^{(1)}e_i \otimes f_j), R_{J^{(1)}e_k}(J^{(1)}e_{i'} \otimes f_{j'})) \\ &= \sum_{k=1}^{\infty}((J^{(1)}e_i, J^{(1)}e_k)f_j, (J^{(1)}e_{i'}, J^{(1)}e_k)f_{j'}) \\ &= \sum_{k=1}^{\infty}((e_k, e_i)f_j, (e_k, e_{i'})f_{j'}) \\ &= \delta_{ii'}\delta_{jj'}\end{aligned}$$

for $i, j, i', j' = 1, 2, \cdots$.

Therefore, $(r(\mathcal{M}^{(1)} \otimes \mathcal{M}^{(2)})r^{-1}$, $HS(\mathcal{H}^{(1)}, \mathcal{H}^{(2)})$, $r(J^{(1)} \otimes J^{(2)})r^{-1}$, $CPHS(\mathcal{H}^{(1)+'}$, $\mathcal{H}^{(2)+}))$ is a standard form. Using the Radon-Nikodym theorem for L^2-spaces [**S**, Theorem 1.2], we obtain the following theorem:

(3.1). *Let $(\mathcal{M}, \mathcal{H}, \mathcal{H}_n^+)$ be a matrix ordered standard form. Then $(r(\mathcal{M}' \otimes \mathcal{M})r^{-1}$, $HS(\mathcal{H}, \mathcal{H})$, $r(J \otimes J)r^{-1}$,$CPHS(\mathcal{H}^+, \mathcal{H}^+))$ is a standard form which is isomorphic to $(\mathcal{M}' \otimes \mathcal{M}, \mathcal{H} \otimes \mathcal{H}, J \otimes J, \mathcal{H}^+ \otimes \mathcal{H}^+)$ by the identification $r : \mathcal{H} \otimes \mathcal{H} \to HS(\mathcal{H}, \mathcal{H})$defined as above. If $A, B \in HS(\mathcal{H}, \mathcal{H})$satisfies $O \trianglelefteq_{cp} A \trianglelefteq_{cp} B$, then there exists an element $C \in \mathcal{M}' \otimes \mathcal{M}$with $O \leq C \leq I$ such that $A = r\hat{C}r^{-1}B$.*

(3.2). *If in (3.1) $\mathcal{M}$ is an injective factor (or semi-finite injective von Neumann algebra) on a separable Hilbert space $\mathcal{H}$, then the above statement is valid for $A \in L(\mathcal{H})$ instead of $A \in HS(\mathcal{H}, \mathcal{H})$.*

Proof. Suppose that $\mathcal{M}$ is the von Neumann algebra in the statement. There then exists an increasing net $\{E_i\}$ of completely positive projections of finite rank on $\mathcal{H}$ which converges strongly to 1 by [**M1**, Theorem 1.4]. It follows that $O \trianglelefteq_{cp} E_iA \trianglelefteq_{cp} E_iB$. Hence

$$\mathrm{Tr}(A^*E_iA) \leq \mathrm{Tr}(B^*E_iB) \leq \mathrm{Tr}(B^*B).$$

Considering a limit with respect to i, we have $\mathrm{Tr}(A^*A) < +\infty$. Using (3.1) we obtain the desired result. □

(3.3). *For a matrix ordered standard form $(\mathcal{M}, \mathcal{H}, \mathcal{H}_n^+)$, any element $A \in HS(\mathcal{H})$ can be uniquely decomposed into the following:*

$$A = A_1 - A_2 + i(A_3 - A_4)$$

where $A_1 \perp A_2, A_3 \perp A_4, A_i \in CPHS(\mathcal{H}^+)$.

The proof of the above proposition is immediate from a decomposition theorem of vectors in the ordered Hilbert space.

§4. Decomposition of positive maps

The purpose of this section is to show that any order isomorphism between non-commutative L^2-spaces associated with von Neumann algebras is decomposed into a sum of a completely positive and a completely co-positive maps. The result is an L^2 version of a theorem of Kadison [**K**] for a Jordan isomorphism on operator algebras.

We first generalize a theorem of A. Connes [**C**] for the polar decomposition of an order isomorphism, to the case where a von Neumann algebra is non-σ-finite.

(4.1). *Let $(\mathcal{M}, \mathcal{H}, J, \mathcal{H}^+)$ and $(\tilde{\mathcal{M}}, \tilde{\mathcal{H}}, \tilde{J}, \tilde{\mathcal{H}}^+)$ be standard forms, and A be a linear bijection of $\mathcal{H}$ onto $\tilde{\mathcal{H}}$ satisfying $A\mathcal{H}^+ = \tilde{\mathcal{H}}^+$. Then for a polar decomposition $A = U|A|$ of A we obtain the following properties:*

(1) *There exists a unique invertible operator B in $\mathcal{M}^+$ such that $|A| = BJBJ$.* (cf. [**I**, Corollary II.3.2])

(2) *There exists a unique Jordan $*$-isomorphism α of $\mathcal{M}$ onto $\tilde{\mathcal{M}}$ such that*

$$(\alpha(X)\xi, \xi) = (XU^{-1}\xi, U^{-1}\xi)$$

for all $X \in \mathcal{M}, \xi \in \tilde{\mathcal{H}}^+$.

Proof. (1) Let $\mathcal{M}$ be non-σ-finite. Choose an increasing net $\{p_i\}_{i \in I}$ of σ-finite projections in $\mathcal{M}$ converging strongly to 1. Put $q_i = p_iJp_iJ$. By [[**C**, Theorem 4.2] $q_i\mathcal{H}^+$ is a closed face of $\tilde{\mathcal{H}}^+$. Since A is an order isomorphism, $A(q_i\mathcal{H}^+)$ is a closed face of $\tilde{\mathcal{H}}^+$. There then exists a σ-finite projection $p_i' \in \tilde{\mathcal{M}}$ such that $A(q_i\mathcal{H}^+) = q_i'\tilde{\mathcal{H}}^+$ where q_i' denotes $p_i'Jp_i'J$. Hence $q_i'Aq_i$ is an order isomorphism of $q_i\mathcal{H}^+$ onto $q_i'\tilde{\mathcal{H}}^+$. These cones appear respectively in the reduced standard forms $(q_i\mathcal{M}q_i,\ q_i\mathcal{H},\ q_iJq_i,\ q_i\mathcal{H}^+)$ and $(q_i'\tilde{\mathcal{M}}q_i',\ q_i'\tilde{\mathcal{H}},\ q_i'Jq_i',\ q_i'\tilde{\mathcal{H}}^+)$. Put $A_i = (q_i'Aq_i)^*q_i'Aq_i$. Then $A_i \in q_i\mathcal{M}^+q_i$ is an order automorphism on $q_i\mathcal{H}^+$. By [**C**, Theorem 3.3] there exists a unique invertible operator $B_i \in q_i\mathcal{M}^+q_i$ such that $A_i = B_iJ_iB_iJ_i$, where J_i denotes q_iJq_i. Taking a logarithm of both sides, we have $\log A_i = \log B_i + J_i(\log B_i)J_i$. Since $\{A_i\}$ is a bounded net, $\{\log B_i\}$ is bounded. Indeed, we have in a standard form that a map

$$X \mapsto \delta_X = \frac{1}{2}(X + JXJ)$$

is a Jordan isomorphism of a selfadjoint part of $\mathcal{M}$ into a selfadjoint part of a set of all order derivations $D(\mathcal{H}^+)$ by [**I**, Corollary VI.2.3]. It is known that any isomorphism of a JB-algebra into another JB-algebra is isometry(see [**HS**, Proposition 3,4.3]). Hence

$$\| \delta_X \| = \| X \|, \quad X \in \mathcal{M}_{\mathrm{s.a.}}.$$

Thus $\{\log B_i\}$ is bounded. It follows that $\{p_i(\log B_i)p_i\}$ is bounded because $p_i\mathcal{M}p_i$ and $q_i\mathcal{M}q_i$ are $*$-isomorphic. Therefore, one can find a subnet of $\{p_i \log B_i p_i\}$ which converges to some element $C \in \mathcal{M}^+$ in the σ-weak topology. We may index the subnet as the same $i \in \mathbf{I}$. We then have for $\xi, \eta \in \mathcal{H}$

$$\begin{aligned}
((C + JCJ)q_j\xi, q_j\eta) &= \lim_i((p_i(\log B_i)p_i + Jp_i(\log B_i)p_iJ)q_j\xi, q_j\eta) \\
&= ((\log B_j + J_j(\log B_j)J_j)q_j\xi, q_j\eta) \\
&= \lim_i(\log A_iq_j\xi, q_j\eta) \\
&= (\log A^*Aq_j\xi, q_j\eta),
\end{aligned}$$

using the facts that $q_iXq_iJq_iXq_iJq_i = p_iXp_iJp_iXp_iJq_i$ for all $X \in \mathcal{M}$, and under the strong topology $\{A_i\}$ converges to A^*A; hence $\{q_i(\log A_i)q_i\}$ converges to $\log A^*A$. Since $\bigcup_{i\in\mathbf{I}} q_i\mathcal{H}$ is dense in $\mathcal{H}$, we obtain the equality $C + JCJ = \log A^*A$. Therefore, $e^CJe^CJ = A^*A$. Thus there exists an element $B \in \mathcal{M}^+$ such that $|A| = BJBJ$. Since, in addition, $q_iBq_iJq_iBq_iJq_i = q_i|A|q_i$, one easily sees the invertibility and the unicity of B using the same properties as in the σ-finite case.

(2) From (1) we have $U = AB^{-1}JB^{-1}J$. It follows that U is an isometry satisfying $U\mathcal{H}^+ = \tilde{\mathcal{H}}^+$. Let p_i and q_i be as in (1). There then exists a σ-finite projection $p_i' \in \tilde{\mathcal{M}}$ such that $U(q_i\mathcal{H}^+) = q_i'\tilde{\mathcal{H}}^+$ with $q_i' = p_i'\tilde{J}p_i'\tilde{J}$. Using also [**C**, Theorem 3.3], one can find a unique Jordan $*$-isomorphism α_i of $q_i\mathcal{M}q_i$ onto $q_i'\tilde{\mathcal{M}}q_i'$ such that

$$(\alpha_i(q_iXq_i)\xi,\xi) = (q_iXq_iU^{-1}\xi,U^{-1}\xi)$$

for all $X \in \mathcal{M}, \xi \in q_i'\tilde{\mathcal{H}}^+$. Fixed now $X \in \mathcal{M}_{\text{s.a.}}$. Since $p_i'\tilde{\mathcal{M}}p_i'$ and $q_i'\tilde{\mathcal{M}}q_i'$ are $*$-isomorphic, there exists a unique operator $Y_i \in p_i'\tilde{\mathcal{M}}_{\text{s.a.}}p_i'$ such that $Y_i|_{q_i'\tilde{\mathcal{H}}} = \alpha_i(q_iXq_i)$. Using an isometry between the Jordan algebras, one sees that $\{\alpha_i(q_iXq_i)\}$ is a bounded net, because $\| \alpha_i(q_iXq_i) \| = \| q_iXq_i \| \leq \| X \|, i \in \mathbf{I}$. Thus $\{Y_i\}$ is bounded. We may then say that $\{Y_i\}$ converges to some operator $Y \in \tilde{\mathcal{M}}_{\text{s.a.}}$ in the σ-weak topology. We then have for $\xi \in \tilde{\mathcal{H}}^+$

$$\begin{aligned}(Yq_j'\xi,q_j'\xi) &= \lim_i(Y_iq_j'\xi,q_j'\xi) = \lim_i(\alpha_i(q_iXq_i)q_j'\xi,q_j'\xi)\\ &= \lim_i(q_iXq_iU^{-1}q_j'\xi,U^{-1}q_j'\xi)\\ &= (XU^{-1}q_j'\xi,U^{-1}q_j'\xi).\end{aligned}$$

Taking a limit with respect to j, we obtain

$$(Y\xi,\xi) = (XU^{-1}\xi,U^{-1}\xi)$$

for all $\xi \in \tilde{\mathcal{H}}^+$. It is known that any normal state on the von Neumann algebra $\tilde{\mathcal{M}}$ is represented by a vector state with respect to an element of $\tilde{\mathcal{H}}^+$ (see [**H**, Lemma 2.10 (1)]). Therefore, the above element Y is uniquely determined. Moreover, we have $q_i'Yq_i' = \alpha_i(q_iXq_i)$. It follows that $\{\alpha_(q_iXq_i)\}$ converges to Y in the strong topology. Hence one can define $\alpha(X) = Y$ for all $X \in \mathcal{M}$. It is now immediate that $\alpha(X^2) = \alpha(X)^2$ for all $X \in \mathcal{M}_{\text{s.a.}}$. Considering the inverse order isomorphism U^{-1}, we have $\alpha(\mathcal{M}) = \tilde{\mathcal{M}}$. This completes the proof. □

In the following proposition we deal with a reduced matrix ordered standard form by a completely positive projection.

(4.2). *With $(\mathcal{M},\mathcal{H},\mathcal{H}_n^+)$ a matrix ordered standard form, let E be a completely positive projection on $\mathcal{H}$. Then $(E\mathcal{M}E,\ E\mathcal{H},\ E_n\mathcal{H}_n^+)$ is a matrix ordered standard*

Proof. The statement was shown in [**MN**, Lemma 3] where $\mathcal{M}$ is σ-finite. In the case where $\mathcal{M}$ is not σ-finite, since E is a completely positive projection, there exists a von Neumann algebra $\mathcal{N}$ such that $(\mathcal{N},\ E\mathcal{H},\ E_n\mathcal{H}_n^+)$ is a matrix ordered standard form by [**M2**, Lemma 3]. Hence $E\mathcal{M}|_{E\mathcal{H}} = \mathcal{N}$ and $(E\mathcal{M}E,\ E\mathcal{H},\ E_n\mathcal{H}_n^+)$ is a matrix ordered standard form by using the same discussion as in the proof in [**M3**]. □

Now, we shall state the decomposition theorem for an order isomorphism between non-commutative L^2-spaces.

(4.3). *Let $(\mathcal{M},\mathcal{H},\mathcal{H}_n^+)$ and $(\tilde{\mathcal{M}},\tilde{\mathcal{H}},\tilde{\mathcal{H}}_n^+)$ be matrix ordered standard forms. Suppose that A is a 1-positive map of $\mathcal{H}$ into $\tilde{\mathcal{H}}$ such that $A\mathcal{H}^+$ is a selfdual cone in the closed range of A. If both the support projection E and the range projection F of A are completely positive, then there exists a centeral projection P of $E\mathcal{M}E$ such that AP is completely positive and $A(E-P)$ is completely co-positive.*

In particular, if A is an order isomorphism of $\mathcal{H}$ onto $\tilde{\mathcal{H}}$, then there exists a centeral projection P of $\mathcal{M}$ such that AP is completely positive and $A(1-P)$ is completely co-positive.

Proof. We first consider the case where A is an order isomorphism. Let U, B and α be as in (4.1). It follows from a theorem of Kadison [**K**] that there exists a central projection P of $\mathcal{M}$ satisfying

$$\alpha : \mathcal{M}_P \to \tilde{\mathcal{M}}_{\alpha(P)},\ \text{onto $*$-isomorphism}$$

and

$$\alpha : \mathcal{M}_{1-P} \to \tilde{\mathcal{M}}_{\alpha(1-P)},\ \text{onto $*$-anti-isomorphism.}$$

Indeed, $\alpha(P)$ is a central projection of $\tilde{\mathcal{M}}$. Since α preserves a $*$-operation and power, $\alpha(P)$ is a projection. Suppose that Q is an arbitrary projection in $\mathcal{M}$. Since α is order preserving, we have $\alpha(QP) \le \alpha(P)$ and $\alpha(Q(1-P)) \le \alpha(1-P)$. It follows that two projections $\alpha(P)$ and $\alpha(QP)$ are commutative, and so are $\alpha(1-P)$ and $\alpha(Q(1-P))$. Hence$\alpha(Q) = \alpha(QP + Q(1-P))$ and $\alpha(P)$ commute. Since α is bijective, a set $\alpha(Q)$ generates a von Neumann algebra $\tilde{\mathcal{M}}$. Therefore, $\alpha(P)$ belongs to a center of $\tilde{\mathcal{M}}$. Now, there then exists a unique completely positive isometry $u : P\mathcal{H} \to \alpha(P)\tilde{\mathcal{H}}$ such that

$$u(P\mathcal{H}^+) = \alpha(P)\tilde{\mathcal{H}}^+) \quad \text{and} \quad \alpha(x) = uxu^{-1}, \quad x \in \mathcal{M}_P$$

by [**M3**, Proposition 2.4] which is also valid for the non-σfinite case. Hence $(UxU^{-1}\xi,\xi) = (uxu^{-1}\xi,\xi), x \in \mathcal{M}_P, \xi \in \alpha(P)\tilde{\mathcal{H}}^+$. We have from the unicity of a completely positive isometry $UP = u$. Note that $\alpha(P)UP = UP$. Indeed, we have for $\xi, \in \alpha(1-P)\tilde{\mathcal{H}}^+$ the equality

$$\| PU^{-1}\xi \|^2 = (UPU^{-1}\xi,\xi) = (\alpha(P)\xi,\xi) = 0.$$

This yields $PU^{-1}\alpha(1-P) = O$, and so $PU^{-1} = PU^{-1}\alpha(P)$. Therefore, we obtain that $AP = UBJBJP = uBJBJP$ and AP is completely positive.

We next consider a $*$-isomorphism $\alpha' : \mathcal{M}_{1-P} \to \tilde{\mathcal{M}}'_{1-\alpha(P)}$ defined by $\alpha'(X) = \tilde{J}\alpha(X)^*\tilde{J}, X \in \mathcal{M}_{1-P}$. There then exists a unique completely positive isometry $v : (1-P)\mathcal{H} \to \alpha(1-P)\tilde{\mathcal{H}}$ such that

$$v(1-P)\mathcal{H}^+ = (1-\alpha(P))\tilde{\mathcal{H}}^+ \quad \text{and} \quad \alpha'(x) = vxv^{-1}, \quad x \in \mathcal{M}_{1-P}.$$

Then we have $\alpha(x) = \tilde{J}vx^*v^{-1}\tilde{J}, x \in \mathcal{M}_{1-P}$. Note that the complete positivity above means $v_n(1-P)_n\mathcal{H}_n^+ = (1-\alpha(P))_n\tilde{\mathcal{H}}_n^{+\prime}$, where $\tilde{\mathcal{H}}_n^{+\prime}$ denotes the selfdual cones associated with $\tilde{\mathcal{M}}'$. Hence v is a completely co-positive map under the setting $(\mathcal{M},\mathcal{H},\mathcal{H}_n^+)$ and $(\tilde{\mathcal{M}},\tilde{\mathcal{H}},\tilde{\mathcal{H}}_n^+)$. Hence

$$\begin{aligned}(UxU^{-1}\xi,\xi) &= (\tilde{J}vx^*v^{-1}\tilde{J}\xi,\xi)\\ &= (\tilde{J}\xi, vx^*v^{-1}\tilde{J}\xi)\\ &= (vxv^{-1}\xi,\xi)\end{aligned}$$

for all $x \in \mathcal{M}_{1-P}, \xi, \in (1-P)\mathcal{H}^+$. It follows that $U(1-P) = v$. We conclude by the equality $A(1-P) = vBJBJ(1-P)$ that $A(1-P)$ is completely co-positive.

We now consider a general A. Since $A\mathcal{H}^+ \subset \tilde{\mathcal{H}}^+$, we have $A\mathcal{H}^+ \subset F\tilde{\mathcal{H}}^+$. Since F is a projection, $F\tilde{\mathcal{H}}^+$ is a selfdual cone in $F\tilde{\mathcal{H}}$. It follows from the selfduality of $A\mathcal{H}^+$ that $A\mathcal{H}^+ = F\tilde{\mathcal{H}}^+$. This yields from (4.2) that FAE is an order isomorphism of $E\mathcal{H}$ onto $F\tilde{\mathcal{H}}$ in the sense of matrix ordered standard forms ($E\mathcal{M}E$, $E\mathcal{H}$, $E_n\mathcal{H}_n^+$)

and $(F\tilde{\mathcal{M}}F,\ F\tilde{\mathcal{H}},\ F_n\tilde{\mathcal{H}}_n^+)$. Using the first part of the proof, we obtain the desired result. Indeed, there exists a central projection $P \in E\mathcal{M}E$ such that FAP is completely positive and $FA(E-P)$ is completely co-positive under the reduced matrix ordered standard forms. We obtain the inclusion

$$^t(A_n(E_n - P_n)\mathcal{H}_n^+) = \ ^t(F_nA_n(E_n - P_n)\mathcal{H}_n^+) \subset F_n\tilde{\mathcal{H}}_n^+ \subset \tilde{\mathcal{H}}_n^+.$$

This completes the proof. □

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